1. $L[3te^{3t}]$

$= 3L[te^{3t}]$

Let $te^{3t} = F(t)$

$L[F(t)] = \int_0^\infty e^{-st} F(t)\, dt$

$$L[t e^{3t}] = \int_0^\infty e^{-st} t e^{3t} \, dt$$

$$= \int_0^\infty \underset{II}{e^{t(3-s)}} \underset{I}{t} \, dt$$

$$= \left[t \frac{e^{t(3-s)}}{3-s} \right]_0^\infty - \int_0^\infty \frac{e^{t(3-s)}}{3-s} \, dt$$

$$= 0 - \frac{1}{3-s} \int_0^\infty e^{t(3-s)} \, dt$$

$$= -\frac{1}{3-s} \left[\frac{e^{t(3-s)}}{3-s} \right]_0^\infty$$

$$= \frac{-1}{3-s} \left(0 - \frac{1}{3-s} \right) = \frac{1}{(3-s)^2}$$

hence

$$3L[t e^{3t}] = \frac{3}{(3-s)^2}$$

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(2) $L[e^{-2t}\cos 4t]$

Let $\cos 4t = F(t)$

$f(s) = L[F(t)] = \cancel{\int_0^\infty e^{-st} F(t)\,dt}$

$= L[\cos 4t] = \cancel{\int_0^\infty e^{-st}\cos 4t\,dt}$

$f(s) = \dfrac{s}{s^2+16}$

$$L[e^{at}F(t)] = f(s-a)$$

$$L[e^{-2t}\cos 4t] = f(s+2)$$

$$= \frac{s+2}{(s+2)^2+16}$$

$$= \frac{s+2}{s^2+4s+20}$$

(3) $L[e^{-2t}(\sinh 2t + 3\cosh 2t)]$

Let $F(t) = \sinh 2t + 3\cosh 2t$

$$L[F(t)] = L[\sinh 2t + 3\cosh 2t]$$

$$= L[\sinh 2t] + 3L[\cosh 2t]$$

$$= \frac{2}{s^2-4} + 3\left(\frac{s}{s^2-4}\right)$$

$$f(s) = \frac{3s+2}{s^2-4}$$

$$L[e^{at}F(t)] = f(s-a)$$

$$L[e^{-2t}(\sinh 2t + 3\cosh 2t)] = \frac{3(s+2)+2}{(s+2)^2 - 4}$$

$$= \frac{3s+8}{s^2+4s}$$

(1) $L[te^t]$

Let $F(t) = e^t$

$L[F(t)] = L[e^t] = \frac{1}{s-1} = f(s)$

then

$L[tF(t)] = (-1)\frac{d}{ds}f(s)$

$= (-1)\frac{d}{ds}\frac{1}{s-1}$

$= (-1)\left[\frac{(s-1)0 - 1(1-0)}{(s-1)^2}\right]$

$= \frac{1}{(s-1)^2}$

(2) $L[t\cos t]$

Let $\cos t = F(t)$

$L[F(t)] = f(s) = L[\cos t]$

$\Longrightarrow f(s) = \frac{s}{s^2+1}$

Soln: $L[tF(t)] = (-1)\frac{d}{ds}f(s)$

$= (-1)\frac{d}{ds}\left(\frac{s}{s^2+1}\right)$

$= -1\left[\frac{(s^2+1)1 - s(2s+0)}{(s^2+1)^2}\right]$

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$$= -1\left[\frac{s^2+1-2s^2}{(s^2+1)^2}\right]$$

$$= \frac{s^2-1}{(s^2+1)^2}$$

(8) $\int_0^\infty t e^{-3t} \sin t \, dt = \frac{3}{50}$

Proof

$L[t \sin t] =$

Let $F(t) = \sin t$

$L[F(t)] = L[\sin t] = f(s)$

$\Rightarrow f(s) = \frac{1}{s^2+1}$

Since $L[t F(t)] = (-1) \frac{d}{ds} f(s)$

$L[t \sin t] = (-1) \frac{d}{ds} \frac{1}{s^2+1}$

$= (-1) \left[\frac{(s^2+1)0 - 1(2s+0)}{(s^2+1)^2} \right]$

$= \frac{2s}{(s^2+1)^2}$

hence

$L[t \sin t] = \int_0^\infty e^{-st} t \sin t \, dt = \frac{2s}{(s^2+1)^2}$

putt $s = 3$

$\int_0^\infty e^{-3t} t \sin t \, dt = \frac{2 \times 3}{(3^2+1)^2} = \frac{6}{100} = \frac{3}{50}$

Proved

Ex $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$ where a is parameter

To evaluate above invers Laplace transform we consider function

$$f(s) = \frac{s}{(s^2+a^2)}$$

then $\frac{d}{da} f(a) = \frac{d}{da}\left[\frac{s}{(s^2+a^2)}\right]$

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$$= s \frac{d}{da} (s^2+a^2)^{-1}$$

$$= s(-1)(2a)(s^2+a^2)^{-2}$$

$$= \frac{-2as}{(s^2+a^2)^2}$$

$$\frac{d}{da} f(s) = -2a \left[\frac{s}{(s^2+a^2)^2}\right]$$

OR $-\frac{1}{2a} \frac{d}{da} f(s) = \frac{s}{(s^2+a^2)^2}$

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = L^{-1}\left[-\frac{1}{2a} \frac{d}{da} f(s)\right]$$

$$= -\frac{1}{2a} L^{-1}\left[\frac{d}{da} f(s)\right]$$

$$= -\frac{1}{2a} \frac{d}{da} L^{-1}[f(s)]$$

$$= -\frac{1}{2a} \frac{d}{da} L^{-1}\left[\frac{s}{s^2+a^2}\right]$$

hence $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = -\frac{1}{2a} \frac{d}{da} \cos at$

$$= -\frac{1}{2a} [-t \sin at]$$

$$= \frac{t \sin at}{2a} \text{ Ans}$$

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(1) $L^{-1}\left[\frac{3s+1}{(s-1)(s^2+1)}\right]$

$$\frac{3s+1}{(s-1)(s^2+1)} = \frac{A}{(s-1)} + \frac{Bs+C}{(s^2+1)}$$

$$3s+1 = A(s^2+1) + (Bs+C)(s-1)$$

$$3s+1 = As^2 + A + Bs^2 - Bs + Cs - C$$

$$A+B=0 \quad \text{—— (1)}$$

$$C-B=3 \quad \text{—— (2)}$$

$$A-C=1 \quad \text{—— (3)}$$

put $s=1$

$A=2$

hence $B=-2$

$C=1$

$$L^{-1}\left[\frac{2}{s-1} + \frac{-2s+1}{(s^2+1)}\right]$$

$$L^{-1}\left[\frac{2}{s-1}\right] - 2L^{-1}\left[\frac{s}{s^2+1}\right] + L^{-1}\left[\frac{1}{s^2+1}\right]$$

$2e^t \cancel{} - 2\cos t + \sin t$ Ans

(1) Using the concept of Laplace transform of convolution of two functions evaluate

$$L^{-1}\left[\frac{1}{(s-2)(s-3)}\right]$$

Solution: We know that

$$L^{-1}[f(s)\cdot g(s)] = F(t) * G(t)$$

$$= \int_0^t F(u)\, G(t-u)\, du$$

Let $f(s) = \frac{1}{s-2}$

then $L^{-1}[f(s)] = e^{2t} = F(t)$

and $g(s) = \frac{1}{s-3}$

$G(t) = L^{-1}[g(s)] = L^{-1}\left[\frac{1}{s-3}\right] = e^{3t}$

hence $L^{-1}[f(s)\cdot g(s)] =$

$$\int_0^t e^{2u}\, e^{3(t-u)}\, du$$

$$e^{3t}\int_0^t e^{-u}\, du$$

$$e^{3t}\left[-e^{-u}\right]_0^t$$

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$e^{3t}\left[-e^{-t}+1\right]$

$-e^{-2t}+1$ Ans

(i) Ordinary linear differential eqn with constant cofficent:

Let us consider an example

Q.N: (5) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 4\sin 2t$

$y(0) = 1$

$y'(0) = 0$

Solution :- The given differential eqn is

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 4y(t) = 4\sin 2t$$

$$y''(t) + 4y'(t) + 4y(t) = 4\sin 2t$$

taking Laplance transform in both the sides we get

$$L[y''(t)] + 4L[y'(t)] + 4L[y(t)] = 4L[\sin(2t)]$$

$$s^2 y(s) - s y(0) - y'(0) + 4[s y(s) - y(0)] + 4 y(s) = 4 \cdot \frac{2}{s^2+4}$$

by given condition

$$[s^2 y(s) - s \cdot 1 - 0] + 4[s y(s) - 1] + 4 y(s) = \frac{8}{s^2+4}$$

$$s^2 y(s) - s + 4 s\, y(s) - 4 + 4 y(s) = \frac{8}{s^2+4}$$

$$y(s)[s^2 + 4s + 4] = \frac{8}{s^2+4} + s + 4$$

$$y(s) = \frac{8}{(s^2+4)(s+2)^2} + \frac{s}{(s+2)^2} + \frac{4}{(s+2)^2}$$

taking inverse laplance transform we get

$$y(t) = 8L^{-1}\left[\frac{1}{(s^2+4)(s+2)^2}\right] + L^{-1}\left[\frac{s}{(s+2)^2}\right] + 4L^{-1}\left[\frac{1}{(s+2)^2}\right]$$

$$y(t) = 8L^{-1}\left[\frac{1}{(s^2+4)(s+2)^2}\right] + L^{-1}\left[\frac{s+4}{(s+2)^2}\right]$$

Now

$$\frac{1}{(s^2+4)(s+2)^2} = \frac{A}{(s+2)} + \frac{B}{(s+2)^2} + \frac{(Cs+D)}{(s^2+4)}$$

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$$1 = A(s+2)(s^2+4) + B(s^2+4) + (Cs+D)(s+2)^2$$

$$1 = A[s^3+4s+2s^2+8] + B(s^2+4) + Cs^3+4Cs+4Cs^2$$
$$+Ds^2+4Ds+4D$$

$A+C=0$ put $s=-2$

$2A+B+4C+D=0$ $B = 1/8$

$4A+4C+4D=0$

$8A+4B+4D=1$

solving above we get

$A = 1/16$

$C = -1/16$

$D = 0$

hence

$$L^{-1}\left[\frac{1}{(s^2+4)(s+2)^2}\right] = \frac{1}{16}L^{-1}\left[\frac{1}{s+2}\right] + \frac{1}{8}L^{-1}\left[\frac{1}{(s+2)^2}\right]$$
$$-\frac{1}{16}L^{-1}\left[\frac{s}{s^2+4}\right]$$

then

$$Y(t) = \frac{8}{16}L^{-1}\left[\frac{1}{s+2}\right] + \frac{8}{8}L^{-1}\left[\frac{1}{(s+2)^2}\right]$$
$$+L^{-1}\left[\frac{1}{s+2}\right] + 2L^{-1}\left[\frac{1}{(s+2)^2}\right] - \frac{8}{16}L^{-1}\left[\frac{s}{s^2+4}\right]$$
$$= \frac{3}{2}L^{-1}\left[\frac{1}{s+2}\right] + 3L^{-1}\left[\frac{1}{(s+2)^2}\right] - \frac{1}{2}L^{-1}\left[\frac{s}{s^2+4}\right]$$

$= \frac{3}{2} e^{-2t} + 3e^{-2t} L^{-1}\left[\frac{1}{s^2}\right] - \frac{1}{2} \cos 2t$

$= \frac{3}{2} e^{-2t} + 3t e^{-2t} - \frac{1}{2} \cos 2t$

$= \left(\frac{3}{2} + 3t\right) e^{-2t} - \frac{1}{2} \cos 2t$ Ans